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The exact solution of the Riemann problem in relativistic magnetohydrodynamics with tangential magnetic fields

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We have extended the procedure to find the exact solution of the Riemann problem in relativistic hydrodynamics to a particular case of relativistic magnetohydrodynamics in which the magnetic field of the initial states is tangential to the discontinuity and orthogonal to the flow velocity. The wave pattern produced after the break up of the initial discontinuity is analogous to the non-magnetic case and we show that the problem can be understood as a purely relativistic hydrodynamical problem with a modified equation of state. The new degree of freedom introduced by the non-zero component of the magnetic field results in interesting effects consisting in the change of the wave patterns for given initial thermodynamical states, in a similar way to the effects arising from the introduction of tangential velocities. Secondly, when the magnetic field dominates the thermodynamical pressure and energy, the wave speeds approach the speed of light, leading to fast shocks and fast and arbitrarily thin rarefaction waves. Our approach is the first non-trivial exact solution of a Riemann problem in relativistic magnetohydrodynamics and it can also be of great interest to test numerical codes against known analytical or exact solutions.

1. Introduction

The decay of a discontinuity separating two constant initial states (Riemann problem) has played a very important role in the development of numerical codes for classical (Newtonian) hydrodynamics after the pioneering work of Godunov (1959). Nowadays, most modern high-resolution shock-capturing methods (LeVeque 1992) are based on the exact or approximate solution of Riemann problems between adjacent numerical cells, and the development of efficient Riemann solvers has become a research field in numerical analysis on its own (see, e.g. Toro 1997). The success of high-resolution shock-capturing methods in many areas of computational fluid dynamics has triggered their extension to classical magnetohydrodynamics (MHD) (e.g. Brio & Wu 1988; for an up-to-date discussion of the issue, see Balsara 2004).

As in other fields in physics, during the last two decades astrophysics, relativity and cosmology have become computational sciences. Modelling and understanding fluid dynamics in astrophysical scenarios is now a key part in research projects involving supernovae, relativistic jets, neutron star instabilities, or accretion onto compact objects, who share a common distinctive feature: either special or general relativity effects are relevant. With this motivation, Riemann solvers have been used in numerical relativistic hydrodynamics since the beginning of the 1990s (Martí, Ibáñez & Miralles 1991).

At present, the use of high-resolution shock-capturing methods based on Riemann solvers is considered the best strategy to solve the equations of relativistic hydrodynamics which has caused the rapid development of Riemann solvers for both special and general relativistic hydrodynamics (see, e.g. Font 2003; Martí & Müller 2003).

The finding by Martí & Müller (1994) of the analytical solution for initial states where the flow is normal to the initial discontinuity boosted the efforts to develop exact Riemann solvers for relativistic hydrodynamics. Pons, Martí & Müller (2000) extended the domain of solutions to problems with arbitrary initial velocities. Rezzolla & Zanotti (2001), for purely normal flow, and Rezzolla, Zanotti & Pons (2003), for the general case, proposed a new procedure to find the solution of the Riemann problem that uses the relativistically invariant relative velocity between the unperturbed initial states. However, to date, no analytical or exact solution of the equations of the Riemann problem in relativistic magnetohydrodynamics has been derived.

The equations of both classical and relativistic magnetohydrodynamics form a non-strictly hyperbolic system. A consequence of the non-strict hyperbolicity of the MHD are the degeneracies in the wave speeds (that lead, e.g. to compound waves, admisible solutions of the planar MHD that involve intermediate shocks), which must be handled analytically with care and hinder the development of both exact and approximate Riemann solvers using the characteristic information for the MHD equations. In the relativistic case, the difficulties in the development of such solvers are increased by the higher nonlinearity of the magnetohydrodynamics system of equations. The characteristic structure of the equations of relativistic magnetohydrodynamics (RMHD) is analysed in Lichnerowicz (1967) and Anile (1989). Approximate Riemann solvers using the characteristic information have been developed by Romero *et al.* (1996), for the same particular magnetohydrodynamic configuration considered here, and by Balsara (2001), Komissarov (1999) and Koldoba, Kuznetsov & Ustyugova (2002), for the general case. A number of analytical solutions involving only shocks, rarefactions and Alfvén waves (Komissarov 1999, 2003) have also been derived.

In this paper, we describe the solution of the Riemann problem for the particular case in which the flow speed has two non-vanishing components and the magnetic field is orthogonal to them. Besides this, we force the flow to have dependence on one spatial coordinate taken along one of the two non-vanishing velocity components. For this particular set-up, the Riemann structure degenerates to only three waves, making the solution attainable. The solution reveals interesting and distinct properties of RMHD and could serve as a guide to the way to the general RMHD Riemann solution.

In this paper, we follow closely the structure and notation used in Pons *et al.* (2000; hereinafter referred to as PMM). The paper is organized as follows. Section 2 collects the relevant equations. Sections 3 and 4 describe, respectively, the flow across rarefactions and shocks setting the ingredients for the Riemann solution, which is discussed in § 5.† Conclusions are drawn in § 6.

2. Equations

Let J^{μ}, $T^{\mu\nu}$ and $F^{*\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) be the components of the density current, the energy–momentum tensor and the Maxwell dual tensor of an ideal magneto-fluid,

† The code computing the exact solution is available on request from the authors. Users of the code can give credit by mentioning the source and citing this paper.

respectively,

$$J^{\mu} = \rho u^{\mu}, \tag{2.1}$$

$$T^{\mu\nu} = \rho \hat{h} u^{\mu} u^{\nu} + \eta^{\mu\nu} \hat{p} - b^{\mu} b^{\nu}, \tag{2.2}$$

$$F^{*\mu\nu} = u^{\mu} b^{\nu} - u^{\nu} b^{\mu}, \tag{2.3}$$

where ρ is the proper rest-mass density, $\hat{h} = 1 + \varepsilon + p/\rho + b^2/\rho$ is the specific enthalpy including the contribution from the magnetic field (b^2 stands for $b^{\mu} b_{\mu}$), ε is the specific internal energy, p the thermal pressure, $\hat{p} = p + b^2/2$ the total pressure, and $\eta^{\mu\nu} = diag(-1, 1, 1, 1)$ the Minkowski metric in Cartesian coordinates. Throughout the paper we use units in which the speed of light is $c = 1$.

The four-vectors representing the fluid velocity and the magnetic field in the fluid rest frame, u^{μ} and b^{μ}, satisfy the conditions $u^{\mu} u_{\mu} = -1$ and $u^{\mu} b_{\mu} = 0$, and there is an equation of state $p = p(\rho, \epsilon)$ that closes the system. All the discussion will be valid for a general equation of state, but results will be shown for an ideal gas, for which $p = (\gamma - 1)\rho\epsilon$, where γ is the adiabatic exponent.

The equations of ideal RMHD correspond to the conservation of rest mass and energy-momentum, and the Maxwell equations. In flat space–time and Cartesian coordinates, these equations read:

$$J^{\mu}{}_{,\mu} = 0, \tag{2.4}$$

$$T^{\mu\nu}{}_{,\mu} = 0, \tag{2.5}$$

$$F^{*\mu\nu}{}_{,\mu} = 0. \tag{2.6}$$

We consider a particular case in which the flow speed has two components and the magnetic field is orthogonal to them. Besides this, we force the flow to have dependence on one spatial coordinate (x) taken along one of the two non-vanishing velocity components. Specifically, we set $u^{\mu} = W(1, v^x, 0, v^z)$, $b^{\mu} = (0, 0, b, 0)$, where W is the flow Lorentz factor. With these restrictions, the above system can be written as a system of conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0, \tag{2.7}$$

where

$$\boldsymbol{U} = (D, \hat{S}^x, \hat{S}^z, \hat{\tau}, B)^T \tag{2.8}$$

is the state vector of conserved quantities and

$$\boldsymbol{F} = (Dv^x, \hat{S}^x v^x + \hat{p}, \hat{S}^z v^x, \hat{S}^x, Bv^x)^T \tag{2.9}$$

is the corresponding vector of fluxes, with

$$D = \rho W, \tag{2.10}$$

$$\hat{S}^i = \rho \hat{h} W^2 v^i \;\; (i = x, z), \tag{2.11}$$

and

$$\hat{\tau} = \rho \hat{h} W^2 - \hat{p} \tag{2.12}$$

being the rest-mass, momentum and total energy densities, and

$$B = bW \tag{2.13}$$

the y-component of the magnetic field as measured in the laboratory frame. Hence, according to these equations, the particular initial configuration chosen together with the imposed symmetry prevent the generation of new components of the velocity and magnetic field.

It is worth noting that for the particular configuration chosen, the term $(b^\mu b^\nu)_{,\mu}$ appearing in the equation of conservation of the stress–energy tensor, vanishes and the RMHD equations reduce to the purely hydrodynamical case with the only contributions from the magnetic field appearing in the pressure and specific enthalpy, and an additional continuity equation for the evolution of the transversal magnetic field. This fact is considered in the Appendix where we explore the possibility of including the magnetic effects of the present configuration in the definition of the equation of state.

According to the previous discussion, the magnetized flow under consideration falls in one of the two degeneracies of the RMHD system (Degeneracy I; Komissarov 1999), for which a description in terms of just three characteristic waves (namely the entropy wave and the two fast magnetosonic waves) is adequate. Turning now towards the solution of the Riemann problem in this particular case, the discontinuity in the initial states breaks down into a couple of left and right propagating rarefaction waves (self-similar continuous flows) and/or shocks and a central tangential discontinuity across which the total pressure, $\hat{p}$, is constant. Both thermal and total pressure increase at fast magnetosonic shocks. Hence, we would use the comparison of the total pressure at the two limiting states to select between shocks and rarefaction waves.

3. Flow across rarefactions

Rarefaction waves are self-similar solutions of the equations that depend on x and t only through the combination $\xi = x/t$. By imposing such a dependence in system (2.7) we obtain the following set of equations

$$(v^x - \xi)\frac{d\rho}{d\xi} + \rho(1 + v^x W^2(v^x - \xi))\frac{dv^x}{d\xi} + \rho W^2 v^z (v^x - \xi)\frac{dv^z}{d\xi} = 0, \quad (3.1)$$

$$W^2\rho\hat{h}(v^x - \xi)\frac{dv^x}{d\xi} + b(1 - v^x\xi)\frac{db}{d\xi} + (1 - v^x\xi)\frac{dp}{d\xi} = 0, \quad (3.2)$$

$$W^2\rho\hat{h}(v^x - \xi)\frac{dv^z}{d\xi} - v^z b\xi\frac{db}{d\xi} - v^z\xi\frac{dp}{d\xi} = 0, \quad (3.3)$$

$$\frac{dp}{d\xi} = hc_s^2\frac{d\rho}{d\xi}, \quad (3.4)$$

$$b(1 + v^x W^2(v^x - \xi))\frac{dv^x}{d\xi} + bW^2 v^z (v^x - \xi)\frac{dv^z}{d\xi} + (v^x - \xi)\frac{db}{d\xi} = 0, \quad (3.5)$$

similar to the system obtained in PMM. The quantity c_s is the sound speed, defined by

$$c_s = \sqrt{\frac{1}{h}\left.\frac{\partial p}{\partial \rho}\right|_s} \quad (3.6)$$

where s is the specific entropy and $h = 1 + \varepsilon + p/\rho$, the specific enthalpy.

Non-trivial similarity solutions exist only if the determinant of system (3.1)–(3.5) vanishes. This leads to the condition

$$\xi^{\pm} = \frac{v^x(1 - \omega^2) \pm \omega\sqrt{(1 - v^2)[1 - v^2\omega^2 - (v^x)^2(1 - \omega^2)]}}{1 - v^2\omega^2}, \quad (3.7)$$

where $\omega^2 = c_s^2 + v_A^2 - c_s^2 v_A^2$ and $v_A^2 = b^2/\rho\hat{h}$ is the Alfvén velocity. The plus and minus signs correspond to rarefaction waves propagating to the left $\mathscr{R}_{\leftarrow}$ and right $\mathscr{R}_{\rightarrow}$, respectively. Note that the values of ξ reduce to those obtained in PMM by replacing ω by c_s (i.e. $b = 0$).

From system (3.1)–(3.5), after some algebraic manipulations, we obtain:

$$W^2\rho\hat{h}(v^x - \xi)\frac{\mathrm{d}v^x}{\mathrm{d}\xi} + b(1 - v^x\xi)\frac{\mathrm{d}b}{\mathrm{d}\xi} + (1 - v^x\xi)\frac{\mathrm{d}p}{\mathrm{d}\xi} = 0, \tag{3.8}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\xi} = hc_s^2\frac{\mathrm{d}\rho}{\mathrm{d}\xi}, \tag{3.9}$$

$$\hat{h}Wv^z = \text{constant}, \tag{3.10}$$

$$\frac{b}{\rho} = \text{constant}. \tag{3.11}$$

Now, using (3.9) and (3.11) to eliminate the differentials of b and ρ, and defining $\hat{\mathscr{B}} = b/\rho$, the ODE (3.8) can be rewritten as

$$\frac{\mathrm{d}v^x}{\mathrm{d}p} = \frac{\left(1 + \hat{\mathscr{B}}^2\rho/hc_s^2\right)}{\rho\hat{h}W^2}\frac{(1 - \xi v^x)}{(\xi - v^x)}, \tag{3.12}$$

in complete analogy with (3.20) in Rezzolla *et al.* (2003) in the case $b = 0$. The analogy can also be extended to the solution procedure. If we define $\hat{\mathscr{A}} = \hat{h}Wv^z$ then, from (3.10),

$$(v^z)^2 = \hat{\mathscr{A}}^2\left(\frac{1 - (v^x)^2}{\hat{h}^2 + \hat{\mathscr{A}}^2}\right), \tag{3.13}$$

which allows us to eliminate the dependence on v^z in (3.7). Now, from the definition of the Lorentz factor, we can derive

$$W^2 = \frac{\hat{h}^2 + \hat{\mathscr{A}}^2}{\hat{h}^2(1 - (v^x)^2)}, \tag{3.14}$$

and, after some algebra,

$$\frac{1 - \xi v^x}{\xi - v^x} = \pm\frac{\sqrt{\hat{h}^2 + \hat{\mathscr{A}}^2(1 - \omega^2)}}{\hat{h}\omega} \tag{3.15}$$

(where ω is the positive root of ω^2). Finally, substituting these last two expressions in (3.12), we obtain

$$\frac{\mathrm{d}v^x}{1 - (v^x)^2} = \pm\frac{\left(1 + \hat{\mathscr{B}}^2\rho/hc_s^2\right)\sqrt{\hat{h}^2 + \hat{\mathscr{A}}^2(1 - \omega^2)}}{\hat{h}^2 + \hat{\mathscr{A}}^2}\frac{\mathrm{d}p}{\rho\omega}. \tag{3.16}$$

The left-hand side of this expression can be integrated analytically and the right-hand side involves only thermodynamical variables and constants. Considering that in a Riemann problem the state ahead of the rarefaction wave is known, the integration of (3.16) allows us to connect the states ahead (a) and behind (b) the rarefaction wave. The normal velocity behind the rarefaction wave can be obtained directly as

$$v_b^x = \tanh\hat{\mathscr{C}}, \tag{3.17}$$

where

$$\hat{\mathscr{C}} = \tfrac{1}{2}\log\left(\frac{1+v_a^x}{1-v_a^x}\right) \pm \int_{p_a}^{p_b} \frac{\left(1+\hat{\mathscr{B}}^2\rho/hc_s^2\right)\sqrt{\hat{h}^2+\hat{\mathscr{A}}^2(1-\omega^2)}}{\hat{h}^2+\hat{\mathscr{A}}^2}\,\frac{\mathrm{d}p}{\rho\omega}. \tag{3.18}$$

The differential of p in the last integral is taken along the adiabats of the equation of state. The isentropic character of rarefaction waves fixes the entropy to that of state a, s_a. Having this in mind, the ODE can be integrated, the solution being only a function of p_b. This can be stated in compact form as

$$v_b^x = \mathscr{R}^a_{\rightleftharpoons}(p_b). \tag{3.19}$$

It is interesting to have an expression for the normal velocity inside the rarefaction wave in terms of the total pressure. This expression can be built taking into account that, from the definition of $\hat{p}$,

$$\mathrm{d}\hat{p} = \mathrm{d}p + \hat{\mathscr{B}}^2\rho\,\mathrm{d}\rho \tag{3.20}$$

and that along an adiabat,

$$\mathrm{d}p = hc_s^2\,\mathrm{d}\rho, \tag{3.21}$$

which combined in result

$$\mathrm{d}\hat{p} = \left(1+\frac{\hat{\mathscr{B}}^2\rho}{hc_s^2}\right)\mathrm{d}p. \tag{3.22}$$

Substitution into (3.18) gives

$$\hat{\mathscr{C}} = \tfrac{1}{2}\log\left(\frac{1+v_a^x}{1-v_a^x}\right) \pm \int_{\hat{p}_a}^{\hat{p}_b} \frac{\sqrt{\hat{h}^2+\hat{\mathscr{A}}^2(1-\omega^2)}}{\hat{h}^2+\hat{\mathscr{A}}^2}\,\frac{\mathrm{d}\hat{p}}{\rho\omega}. \tag{3.23}$$

Note that the previous expression is identical to that derived by Rezzolla *et al.* (2003) in the case of (non-magnetized) relativistic hydrodynamics after removing the hats and substituting ω by the sound speed. The corresponding compact notation for the function giving v_b^x in terms of $\hat{p}_b$ will be

$$v_b^x = \hat{\mathscr{R}}^a_{\rightleftharpoons}(\hat{p}_b). \tag{3.24}$$

Function $\mathscr{R}^a_{\rightleftharpoons}(p)$ is shown in figure 1, for different values of the invariant $\hat{\mathscr{B}}$, the various branches of the curves corresponding to rarefaction waves propagating towards or away from a. Rarefaction waves move towards (away from) a, if the pressure inside the rarefaction is smaller (larger) than p_a. The last assertion also applies for the total pressure, $\hat{p}$. In a Riemann problem the state a is ahead of the wave and only those branches corresponding to waves propagating towards a in figure 1 must be considered. Moreover, we can discriminate between waves propagating towards the left and the right by taking into account that the initial left (right) state can only be reached by a wave propagating towards the left (right). The addition of a transverse magnetic field in the limiting state forces the value of the normal velocity within the rarefaction wave to have larger absolute values. This effect is a consequence of the fact that the absolute value of the slope of the function $\mathscr{R}^a_{\rightleftharpoons}(p)$ at p_a, $|\mathscr{R}^{a\prime}_{\rightleftharpoons}(p_a)|$, is an increasing function of $\hat{\mathscr{B}}$ (or b_a). Moreover, it can be easily proved that

$$|\mathscr{R}^{a\prime}_{\rightleftharpoons}(p_a;\hat{\mathscr{B}}\to\infty)| = \frac{1}{c_{sa}}|\mathscr{R}^{a\prime}_{\rightleftharpoons}(p_a;\hat{\mathscr{B}}=0)|, \tag{3.25}$$

where c_{sa} is the sound speed at state a. Hence, the sound speed at state a limits in

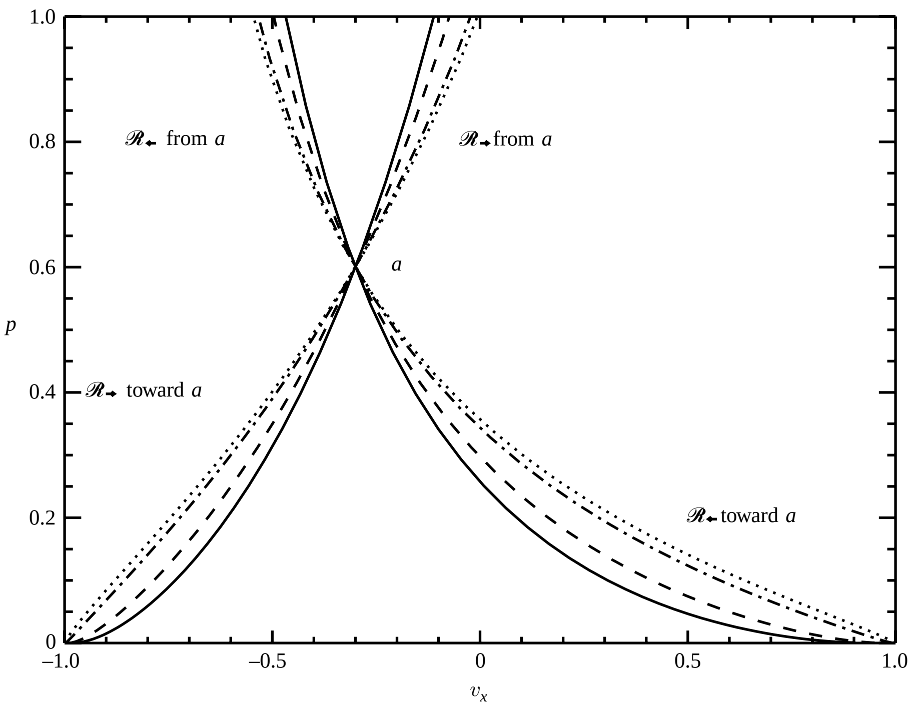

FIGURE 1. Loci of states which can be connected with a given state a by means of relativistic rarefaction waves propagating to the left ($\mathscr{R}_{\leftarrow}$) and to the right ($\mathscr{R}_{\rightarrow}$) and moving towards or away from a. Solutions for $\hat{\mathscr{B}} = 0, 1, 3$ and 10, correspond to solid, dashed, dashed-dotted and dotted lines, respectively. The state a is characterized by $p_a = 0.6$, $\rho_a = 1.0$, and $v_a^x = -0.3$. An ideal gas EOS with $\gamma = 5/3$ was assumed.

practice the range of values of the normal velocity within the rarefaction wave (see figure 1 where the curves corresponding to $\hat{\mathscr{B}} = 3$ and 10 almost coincide). Another consequence of the previous result is that in the extreme case for which c_{sa} tends to the light speed, the presence of a transverse magnetic field in the limiting state would have no practical effect on the rarefaction wave. The effect of the magnetic field must be combined with that coming from the presence of tangential velocities in state a which operates in the opposite direction (see PMM).

4. Jumps across shocks

If Σ is a hyper-surface in Minkowski space–time across which ρ, u^μ, $T^{\mu\nu}$ and $F^{*\mu\nu}$ are discontinuous, the Rankine–Hugoniot conditions are given by (Lichnerowicz 1967; Anile 1989)

$$[\rho u^\mu] n_\mu = 0, \tag{4.1}$$

$$[T^{\mu\nu}] n_\nu = 0, \tag{4.2}$$

$$[F^{*\mu\nu}] n_\nu = 0, \tag{4.3}$$

where n_μ is the unit normal to Σ, and where we have used the notation

$$[G] = G_a - G_b, \tag{4.4}$$

G_a and G_b being the boundary values of G on the two sides of Σ.

Considering Σ as the hyper-surface in four-dimensional space describing the evolution of a shock wave normal to the x-axis, the unitarity of n_ν allows us to write it as

$$n^\nu = W_s(V_s, 1, 0, 0), \tag{4.5}$$

where V_s is interpreted as the coordinate velocity of the surface that defines the position of the shock wave and W_s is the Lorentz factor of the shock,

$$W_s = \frac{1}{\sqrt{1 - V_s^2}}. \tag{4.6}$$

Equations (4.1) and (4.3) allow us to introduce two invariants across the shock

$$j \equiv W_s D_a \left(V_s - v_a^x\right) = W_s D_b \left(V_s - v_b^x\right), \tag{4.7}$$

$$f \equiv W_s B_a \left(V_s - v_a^x\right) = W_s B_b \left(V_s - v_b^x\right), \tag{4.8}$$

where $B = bW$. Quantity j represents the mass flux across the shock and according to our definition, j is positive for shocks propagating to the right (the same convention as that used in Martí & Müller 1994; PMM). Dividing (4.7) by (4.8), we find that the quantity B/D (or, equivalently, b/ρ) is constant across shocks, as it was through rarefaction waves.

Next, the Rankine–Hugoniot conditions (4.1), (4.2) can be written in terms of the conserved quantities D, $\hat{S}^j$ and $\hat{\tau}$, and j as follows

$$[v^x] = -\frac{j}{W_s}\left[\frac{1}{D}\right], \tag{4.9}$$

$$[\hat{p}] = \frac{j}{W_s}\left[\frac{\hat{S}^x}{D}\right], \tag{4.10}$$

$$\left[\frac{\hat{S}^z}{D}\right] = 0, \tag{4.11}$$

$$[v^x \hat{p}] = \frac{j}{W_s}\left[\frac{\hat{\tau}}{D}\right]. \tag{4.12}$$

Now from (4.11) we have that the quantity $\hat{h}Wv^z$ is constant across the shock, as it is through rarefactions.

We note that in deriving equations (4.9)–(4.12) we have made use of the fact that the mass flux is non-zero across a shock. The conditions across a tangential discontinuity imply continuous total pressure and normal velocity (by setting $j = 0$ in equations (4.9), (4.10) and (4.12)), and an arbitrary jump in the tangential velocity and transverse magnetic field.

Our aim now is to write v_b^x, the normal flow speed in the post-shock state, as a function of the post-shock pressure p_b. As a first step, we write v_b^x as a function of $\hat{p}_b$, j and V_s (and the preshock state, a). Given the complete analogy between the jump conditions (4.7), (4.9)–(4.12), and the corresponding expressions in PMM, we write

$$v_b^x = \left(\hat{h}_a W_a v_a^x + \frac{W_s(\hat{p}_b - \hat{p}_a)}{j}\right)\left(\hat{h}_a W_a + (\hat{p}_b - \hat{p}_a)\left(\frac{W_s v_a^x}{j} + \frac{1}{\rho_a W_a}\right)\right)^{-1}. \tag{4.13}$$

The dependence on the magnetic field in the pre- and post-shock states is hidden in the definitions of $\hat{p}$ and $\hat{h}$.

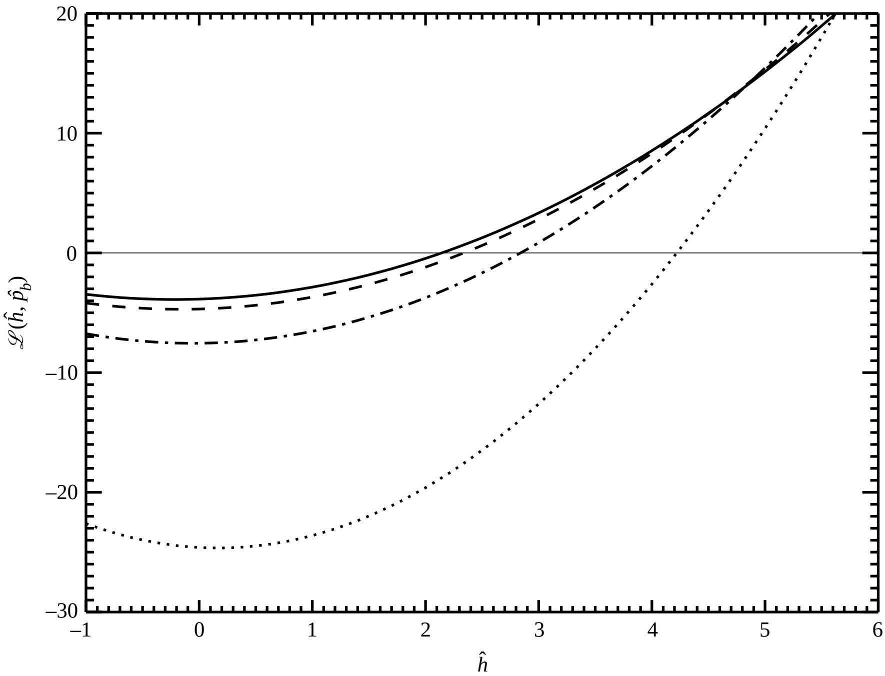

FIGURE 2. Graphical representation of the function $\mathscr{L}^a(\hat{h}; \hat{p}_b)$ (see text for definition) whose zeros for varying post-shock pressures, $\hat{p}_b$, define the Lichnerowicz adiabat. The state a is characterized by $p_a = 0.25$, $\rho_a = 1.0$ and $v_a^x = 0.5$. Solutions for transversal magnetic fields in state a, $\hat{\mathscr{B}}_a = 0, 0.5, 1.0$ and 2.0, correspond to solid, dashed, dashed-dotted and dotted lines, respectively. $\hat{p}_b$ was chosen to be 1.0. An ideal gas EOS with $\gamma = 5/3$ was assumed.

The shock speed V_s can be eliminated using the definition of mass flux to obtain

$$V_s^{\pm} = \frac{\rho_a^2 W_a^2 v_a^x \pm |j| \sqrt{j^2 + \rho_a^2 W_a^2 \left(1 - v_a^{x\,2}\right)}}{\rho_a^2 W_a^2 + j^2}, \tag{4.14}$$

where V_s^+ (V_s^-) corresponds to shocks propagating to the right (left).

Proceeding in the same way as in PMM (i.e. $[T^{\mu\nu}] n_\nu \{(\hat{h} u_\mu)_a + (\hat{h} u_\mu)_b\} = 0$) to derive the Taub adiabat, we can now obtain

$$[\hat{h}^2] = \left(\frac{\hat{h}_b}{\rho_b} + \frac{\hat{h}_a}{\rho_a}\right) [\hat{p}], \tag{4.15}$$

i.e. the Lichnerowicz adiabat (Anile 1989) particularized to our special set-up. Figure 2 represents the function

$$\mathscr{L}^a(\hat{h}; \hat{p}_b) \equiv \hat{h}^2 - \hat{h}_a^2 - \left(\frac{\hat{h}}{\rho(\hat{h}, \hat{p}_b)} + \frac{\hat{h}_a}{\rho_a}\right) (\hat{p}_b - \hat{p}_a) \tag{4.16}$$

for an ideal gas equation of state, although the general shape of the curve (positive asymptotic branches; negative value for $\hat{h} = 1$) is independent of the equation of state. The (unique) root at the right of $\hat{h} = 1$ defines the thermodynamical post-shock state.

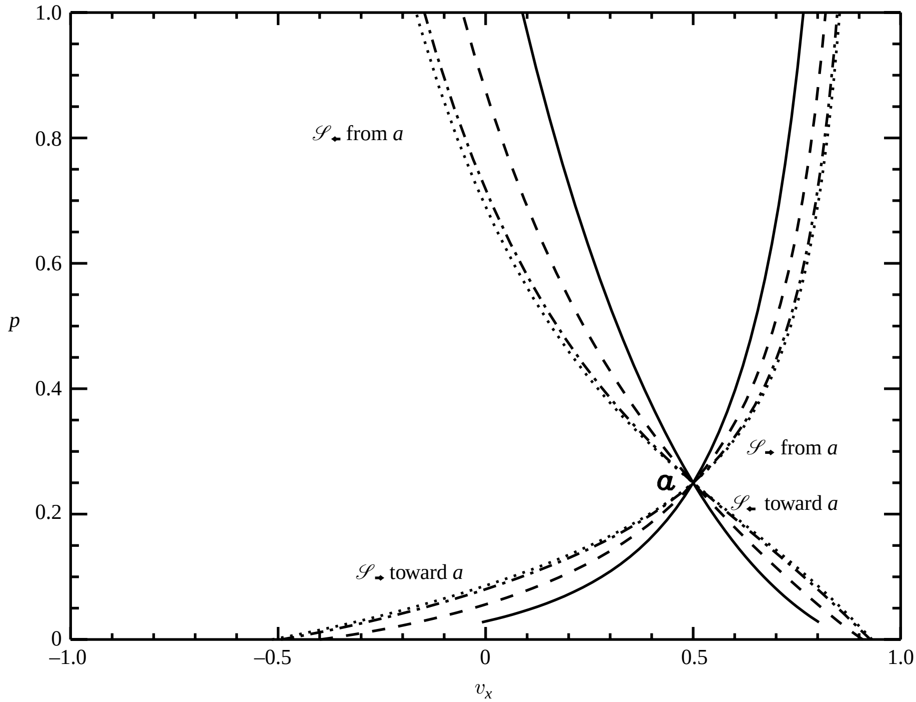

FIGURE 3. Loci of states which can be connected with a given state a by means of relativistic shock waves propagating to the left ($\mathscr{S}_{\leftarrow}$) and to the right ($\mathscr{S}_{\rightarrow}$) and moving towards or away from a. Solutions for $\hat{\mathscr{B}} = 0, 1, 3$ and 10, correspond to solid, dashed, dashed-dotted and dotted lines, respectively. The state a is characterized by $p_a = 0.25$, $\rho_a = 1.0$, and $v_a^x = 0.5$. An ideal gas EOS with $\gamma = 5/3$ was assumed.

Equation (4.15) together with the definitions of $\hat{p}$ and $\hat{h}$, the equation of state and the constancy of b/ρ through the shock, allows us to write ρ_b as a function of p_b and the preshock state a. Next, multiplying (4.2) by n_μ and using the definition of relativistic mass flux, we obtain

$$j^2 = \frac{-[\hat{p}]}{[\hat{h}/\rho]}. \tag{4.17}$$

Using the positive (negative) root of j^2 for shock waves propagating towards the right (left), equation (4.17) allows us to obtain the desired relation between the post-shock normal velocity v_b^x and the post-shock pressure p_b. In a compact form, the relation reads

$$v_b^x = \mathscr{S}^a_{\rightleftharpoons}(p_b). \tag{4.18}$$

Alternatively, the relation can be written as a function of $\hat{p}_b$

$$v_b^x = \hat{\mathscr{S}}^a_{\rightleftharpoons}(\hat{p}_b). \tag{4.19}$$

Let us note that the expressions used to build up the function $\hat{\mathscr{S}}^a_{\rightleftharpoons}$, namely (4.13), (4.14), (4.15) and (4.17), are formally identical to those corresponding to the pure (i.e. non-magnetized) relativistic hydrodynamical case. The difference appears in the definition of the function $\rho = \rho(\hat{h}, \hat{p})$, which leads to different roots of the function $\mathscr{L}^a(\hat{h}; \hat{p}_b)$, (4.16). Function $\mathscr{S}^a_{\rightleftharpoons}(p)$ is shown in figure 3, for different values of the invariant $\hat{\mathscr{B}}$, the various branches of the curves corresponding to shock waves propagating towards or away from a. In order to select the relevant branch of the function $\mathscr{S}^a_{\rightleftharpoons}(p)$, the same argumentation as in the case of rarefaction waves can be used (see § 3). As in the case of rarefaction waves, the addition of a transverse magnetic field in the limiting state forces the value of the normal velocity in the pre-/post-shock state to have larger absolute values. Again, this effect must be combined with that coming from the presence of tangential velocities in state a (see PMM).

Once v^x_b is known, v^z_b can be obtained through

$$\left(v^z_b\right)^2 = \hat{\mathscr{A}}^2\left(\frac{1-\left(v^x_b\right)^2}{\hat{h}^2_b+\hat{\mathscr{A}}^2}\right), \tag{4.20}$$

where we have defined $\hat{\mathscr{A}} = \hat{h}_a W_a v^z_a$. Analogously, $b_b = \hat{\mathscr{B}}\rho_b$ with $\hat{\mathscr{B}} = b_a/\rho_a$.

5. The solution of the Riemann problem

As discussed in § 2, for the particular case under consideration (magnetic field orthogonal to both the fluid velocity and the wave propagation direction), the time evolution of a Riemann problem with initial states L (left) and R (right) can be represented as:

$$LR \;\rightarrow\; L\,\mathscr{W}_{\leftarrow}\,L_*\,\mathscr{C}\,R_*\,\mathscr{W}_{\rightarrow}\,R, \tag{5.1}$$

where $\mathscr{W}$ and $\mathscr{C}$ denote a (fast magnetosonic-) shock or rarefaction, and a contact discontinuity, respectively. The arrows ($\leftarrow$ / $\rightarrow$) indicate the direction (left/right) from which fluid elements enter the corresponding wave.

The solution of the Riemann problem consists in finding the intermediate states, L_* and R_*, as well as the positions of the waves separating the four states (which only depend on L, L_*, R_* and R). The functions $\mathscr{W}_{\rightarrow}$ and $\mathscr{W}_{\leftarrow}$ allow us to determine the functions $v^x_{R*}(\hat{p})$ and $v^x_{L*}(\hat{p})$, respectively. The pressure $\hat{p}_*$ and the flow velocity v^x_* in the intermediate states are then given by the condition

$$v^x_{R*}(\hat{p}_*) = v^x_{L*}(\hat{p}_*) = v^x_*. \tag{5.2}$$

The functions $v^x_{S*}(\hat{p})$ are defined by

$$v^x_{S*}(\hat{p}) = \begin{cases} \hat{\mathscr{R}}^S(\hat{p}) & \text{if } \hat{p} \leqslant \hat{p}_S, \\ \hat{\mathscr{S}}^S(\hat{p}) & \text{if } \hat{p} > \hat{p}_S, \end{cases} \tag{5.3}$$

where $\hat{\mathscr{R}}^S(\hat{p})$ ($\hat{\mathscr{S}}^S(\hat{p})$) denotes the family of all states which can be connected through a rarefaction (shock) with a given state S (L, R) ahead of the wave. Once $\hat{p}_*$ and v^x_* have been obtained, the remaining quantities can be computed.

Figure 4 shows the solution of a particular Riemann problem for different values of $\hat{\mathscr{B}} = 0, 1.0, 2.0, 4.0$ in the initial states. The crossing point of any two lines gives the pressure and the normal velocity in the intermediate states.

It must be noted that the resolution of the Riemann problem under consideration can be formally done in the same way as the pure (relativistic) hydrodynamical problem with some modifications (see the Appendix). First of all, p and h have to be replaced by $\hat{p}$ and $\hat{h}$. Secondly, the sound speed in the integrand of (3.18) has to be replaced by ω. Finally, the equation of state that provides the rest-mass density as a function of the pressure and the enthalpy, $\rho(\hat{h}, \hat{p})$, has to be modified to include the contributions from the magnetic field. Given the parallelism between the present

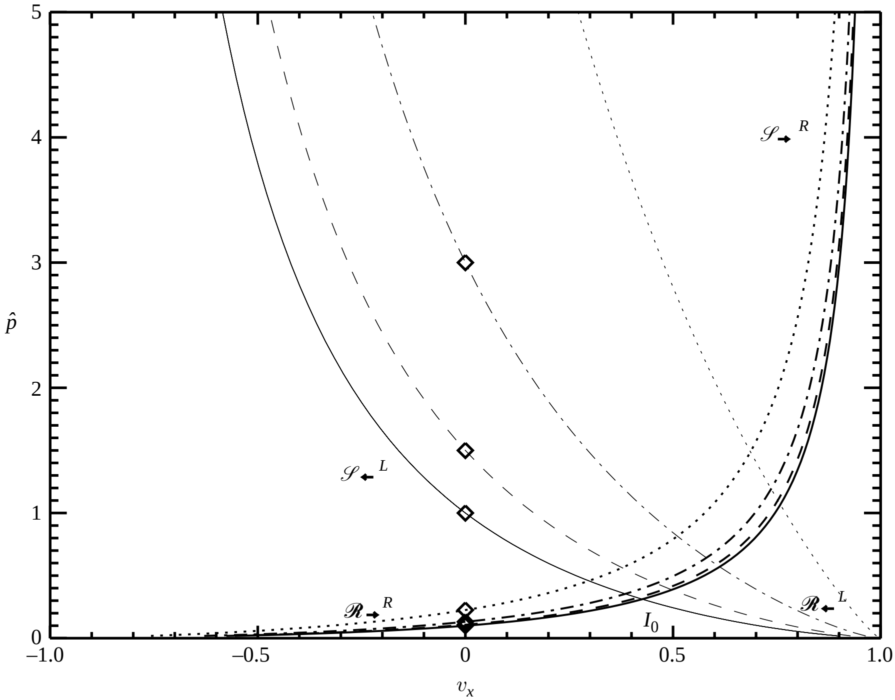

FIGURE 4. Graphical solution in the $(\hat{p}, v^x)$-plane of the relativistic Riemann problem with initial data $p_L = 1.0$, $\rho_L = 1.0$, $v^x_L = 0$; $p_R = 0.1$, $\rho_R = 0.125$ and $v^x_R = 0$ for different values of $\hat{\mathscr{B}} = 0, 1.0, 2.0$ and 4.0, in the left and right states represented by solid, dashed, dashed-dotted and dotted lines, respectively. Diamonds indicate the initial states. An ideal gas EOS with $\gamma = 1.4$ was assumed. The crossing point of any two lines gives the pressure and the normal velocity in the intermediate states. I_0 gives the solution for vanishing magnetic field.

particular (relativistic) magnetohydrodynamical case and the purely hydrodynamical one, the effects concerning the smooth transition from one wave pattern to another when the tangential velocities in the initial states are changed (Rezzolla & Zanotti 2002; Rezzolla *et al.* 2003) will extend to the present case for fixed initial values of the magnetic field. Hence, we concentrate on the effects on the solution induced by varying the initial magnetic fields. Figure 5 shows the solution of a Riemann problem with (*a*) vanishing magnetic field, and (*b*) $b_R = 0.8$. Whereas in the purely hydrodynamical case the Riemann solution gives rise to a left-propagating rarefaction wave and right-propagating contact and shock waves, the case with non-vanishing magnetic field leads to a couple of shock waves and a left-propagating contact discontinuity. The reason for this qualitative change (rarefaction/shock to shock/shock) can be found in the increase of the total pressure in the initial right state of the magnetized case. This increases the total pressure in the intermediate states at the two sides of the contact discontinuity. When this pressure becomes larger than that at the initial left state then a left-propagating shock instead of a rarefaction is produced.

Also noticeable from figure 5 is the increase of velocity of the shock propagating towards the right in the magnetized case. The increase of the velocity of propagation of fast waves for increasing magnetic fields (approaching the light speed in the fluid rest frame for strong enough fields, much larger than equipartition) is well-known in

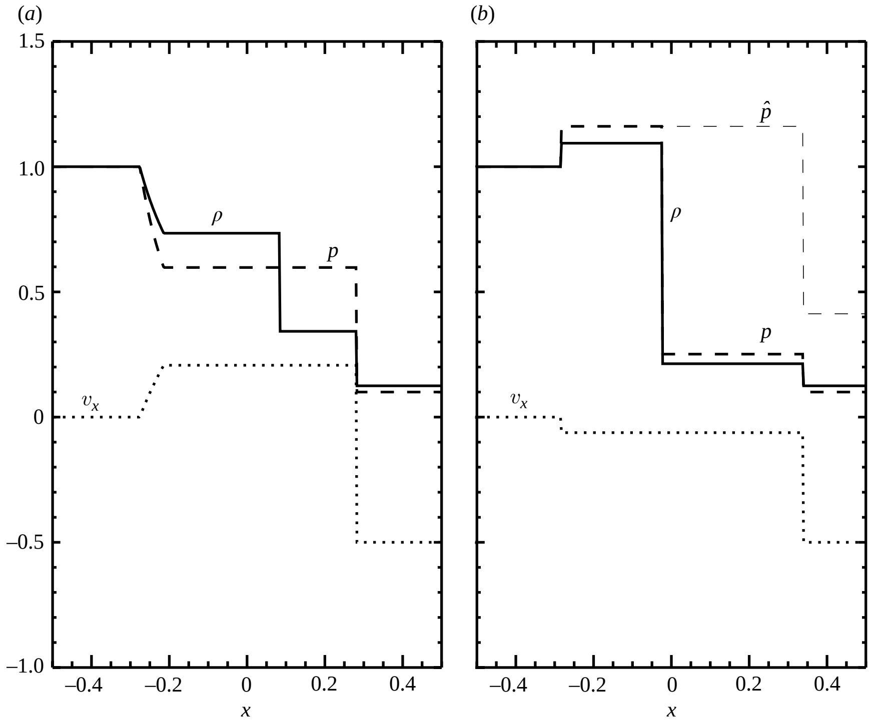

FIGURE 5. Solution of the Riemann problem with initial data $p_L = 1.0$, $\rho_L = 1.0$, $v^x_L = 0.0$; $p_R = 0.1$, $\rho_R = 0.125$ and $v^x_R = -0.5$, at time $t = 0.4$. An ideal gas EOS with $\gamma = 5/3$ was assumed. (*a*) Purely hydrodynamical problem. (*b*) With $b_R = 0.8$. Note that $\hat{p}$ is constant through the contact discontinuity whereas the thermal pressure, p, is not.

RMHD (e.g. Anile 1989). In the particular magnetic problem under consideration, it leads to the increase of velocity of propagation of rarefaction heads and tails, and of shocks, and can be understood as follows. Equation (3.7) gives the propagation speed of the head/tail of a right-/left-propagating rarefaction wave in a given state. Taking v^x, $v = 0$ in (3.7), we obtain $|\xi^{\pm}| = \omega$ and $\omega \rightarrow 1$ for large enough b. Remember that for purely hydrodynamical (relativistic) flows, $|\xi^{\pm}|$ has the sound speed as a limiting value. A similar result holds for shocks. For preshock states at rest, $|V_s^{\pm}| \rightarrow 1$ as the magnetic field in the preshock state is increased. This can be seen by remembering that the shocks that appear in our configurations are super-magnetosonic, and that the fast magnetosonic speed in the preshock state (ω_a) tends to light speed when $b_a \rightarrow \infty$.

Finally, let us note that the drift towards solutions involving only discontinuous waves (shock waves and rarefactions of negligible width) for increasing magnetic fields as concluded in the previous paragraph, is consistent with the fact that in the limit of strong magnetization, the equations of RMHD reduce to the equations of force-free degenerate electrodynamics, whose Riemann problem only involves (linearly degenerate) discontinuous waves (Komissarov 2002).

6. Summary and conclusions

We have obtained an exact solution of the Riemann problem for multidimensional relativistic magnetohydrodynamics in the particular case in which the magnetic field is normal to the fluid velocity. In this particular problem, the complex 7-wave pattern of RMHD is reduced to two fast magnetosonic waves and a contact discontinuity, which allows us to use the same procedure as in the non-magnetic case. Alternatively, we have shown that the problem can be understood as a purely RHD situation with a modified equation of state (see the Appendix for details).

Two interesting features arise from our results. First, for fixed initial thermodynamical states, it is possible to change continuously from one wave pattern to another (shock/shock, shock/rarefaction, rarefaction/rarefaction) analogously to what happens when tangential velocities are introduced (Rezzolla & Zanotti 2002). Secondly, we recover the result for RMHD flows with general magnetic field configurations establishing the tendency of the wave speeds to the light speed when the magnetic field dominates the thermodynamical pressure and energy. For our particular configuration of the magnetic field, this results in fast-moving shock waves and rarefaction waves in which the distance between the head and the tail is progressively reduced as the magnetic field increases. The drift towards solutions involving only discontinuous waves (shock waves and rarefactions of negligible width) for increasing magnetic fields is consistent with the fact that in the limit of strong magnetization, the equations of RMHD reduce to the equations of force-free degenerate electrodynamics, whose Riemann problem only involves discontinuous waves (Komissarov 2002).

In addition to the theoretical interest of our results, an exact solution of the RMHD Riemann problem is relevant for the development of numerical codes. Up to now, in order to test the various algorithms and approximate Riemann solvers developed for numerical applications, one could only increase the spatial resolution and hope that the numerical solution converged to the physical one. Having an exact solution to compare with, even if it is just a particular case, allows for a more rigorous testing and error estimation. Last but not least, it is more convenient to start understanding and solving a simpler case before attempting the solution of the full problem, which is the next natural extension of this work.

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Appendix. A hydrodynamical approach

Equations (2.7) are identical to those for the purely hydrodynamical case by replacing

$$p \longrightarrow \hat{p} = p + \tfrac{1}{2}b^2, \tag{A 1}$$

$$h \longrightarrow \hat{h} = h + \frac{b^2}{\rho}, \tag{A 2}$$

indicating that a description of the present particular RMHD problem based on a purely hydrodynamical approach with a different equation of state may be possible. In this Appendix, we explore such a possibility, first suggested by Romero *et al.* (1996). The key point is to eliminate the magnetic field from the equations by building up a thermodynamically consistent equation of state including the effects of the magnetic field.

It follows from (2.7) that

$$\frac{\mathrm{D}(b/\rho)}{\mathrm{D}t} = 0, \tag{A 3}$$

where $\mathrm{D}/\mathrm{D}t$ stands for the standard convective derivative, implying that the evolution of the fluid elements is along states keeping b/ρ = constant. Then for a particular fluid element, $\hat{p}$ and $\hat{h}$ can be written as

$$\hat{p} = p + \tfrac{1}{2}\hat{\mathscr{B}}^2\rho^2, \tag{A 4}$$

$$\hat{h} = 1 + \varepsilon + \frac{\hat{p}}{\rho} + \tfrac{1}{2}\hat{\mathscr{B}}^2\rho, \tag{A 5}$$

where $\hat{\mathscr{B}}$ is a constant.

Consistency of the two previous expressions with the definition of $\hat{h}$ is fulfilled by defining a new specific internal energy, $\hat{\varepsilon}$,

$$\hat{\varepsilon} = \varepsilon + \tfrac{1}{2}\hat{\mathscr{B}}^2\rho. \tag{A 6}$$

The evolution of the fluid elements in a perfect fluid is adiabatic. Hence now we look for the adiabats of the new equation of state, $\hat{p} = \hat{p}(\rho, \hat{\varepsilon})$. The fact that the evolution of the fluid elements keeps $\hat{\mathscr{B}}$ = constant draw us to consider that $\hat{\mathscr{B}}$ is constant along the adiabats of the new equation of state, $\hat{s}$ = constant. We shall use this to look for the desired relation between the entropies of the two equations of state, s and $\hat{s}$. To do this, we differentiate (A 6) along transformations keeping $\hat{s}$ = constant. We obtain

$$\mathrm{d}\hat{\varepsilon} = \mathrm{d}\varepsilon + \tfrac{1}{2}\hat{\mathscr{B}}^2\,\mathrm{d}\rho. \tag{A 7}$$

On the other hand, according to the first law of thermodynamics, for an adiabatic transformation,

$$\mathrm{d}\hat{\varepsilon} = \frac{\hat{p}}{\rho^2}\,\mathrm{d}\rho. \tag{A 8}$$

Substitution of $\hat{p}$ into the previous expression gives

$$\mathrm{d}\hat{\varepsilon} = \frac{p}{\rho^2}\,\mathrm{d}\rho + \tfrac{1}{2}\hat{\mathscr{B}}^2\,\mathrm{d}\rho. \tag{A 9}$$

Finally, comparison with (A 7) leads to

$$\mathrm{d}\varepsilon = \frac{p}{\rho^2}\,\mathrm{d}\rho, \tag{A 10}$$

which is formally identical to the variation of internal energy in an adiabatic transformation s = constant. Taking into account that the differentials were taken along the adiabats of the new equation of state, the conclusion is that the entropy in the new equation of state must be a function of the entropy in the original equation of state only.

Now the sound speed of the new equation of state,

$$\hat{c}_s = \sqrt{\frac{1}{\hat{h}}\left.\frac{\partial\hat{p}}{\partial\rho}\right|_{\hat{s}}}, \tag{A 11}$$

can be derived. The substitution of $\hat{p}$ following (A 4) and the equivalence of the adiabats of the two equations of state leads to

$$\hat{c}_s = \sqrt{\frac{h}{\hat{h}}c_s^2 + \frac{\hat{\mathscr{B}}^2\rho}{\hat{h}}}, \tag{A 12}$$

where h and c_s stand for the enthalpy and the sound speed of the original equation of state, respectively. Finally, a small amount of algebra allows us to write,

$$\hat{c}_s = \sqrt{\left(1 - v_A^2\right)c_s^2 + v_A^2}, \tag{A 13}$$

where v_A is the Alfvén speed for our particular case in which only one component of the magnetic field is non-zero, $v_A = b/\sqrt{\rho\hat{h}}(= \hat{\mathscr{B}}\sqrt{\rho/(h + \hat{\mathscr{B}}^2\rho)})$. Note that $\hat{c}_s$ coincides with the quantity ω defined in § 2.

Note that although the original equation of state could have a sound speed significantly smaller than light speed (e.g. $\leqslant 1/\sqrt{3}$, for an ultra-relativistic non-degenerate ideal gas), the sound speed of the new equation of state (which represents the true propagation speed of perturbations in our magnetized fluid) approaches the light speed for large enough values of $\hat{\mathscr{B}}$ (or b). Finally, notice that the convexity of the EOS is ensured, since $\mathscr{B}^2$ is a positive defined quantity and therefore $\partial^2\hat{p}/\partial\rho^2|_s > 0$.

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